Games and Economic Behavior 75 (2012) 613–624

Contents lists available at SciVerse ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

What money can't buy: Efficient mechanism design with costly signals

Daniele Condorelli [1]

Department of Economics, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, UK

ARTICLE INFO

Article history:
Received 6 April 2011
Available online 8 March 2012

JEL classification:
D45
D82
H42

Keywords:
Mechanism design
Costly signals
Priority lists
Lotteries

ABSTRACT

I study the ex-ante efficient allocation of a set of quality-heterogeneous objects to a number of heterogeneous risk-neutral agents. Agents have independent private values, which represent the maximum cost they are willing to sustain to obtain an object of unitary quality. The designer faces a trade-off between allocative efficiency and cost of screening, because the cost sustained is wasted. The optimal mechanism ranks agents based on their marginal contribution to social surplus and distributes objects to higher-ranked agents. The ranking is independent of the scarcity of objects or the extent of their heterogeneity. If the hazard rates of the distributions of values are increasing, agents are ranked according to their expected values. If hazard rates are decreasing and agents are symmetric, the objects are allocated to the agents that sustain the highest costs. In general, optimal mechanisms combine both pooling and screening of values.

1. Introduction

Motivation and aim of the paper. There are things that money cannot buy. Scarce medical resources are assigned through priority lists almost everywhere. A fraction of immigration permits in the US are allocated by lottery. When demand exceeds supply, a large number of goods are rationed using waiting lines, rather than by raising their prices to clear the market. More generally, mechanisms that do not involve monetary payments from the prospective recipients are widely adopted for the allocation of public resources.[2]

The problem of how to allocate efficiently a set of indivisible objects to agents with private information when individual valuations represent the willingness to pay and the designer can collect monetary payments from agents has been widely studied.[3] Moreover, there is a substantial body of literature interested in optimal matching of objects to privately informed agents in the absence of money or other (non-cheap) signaling devices.[4] However, a systematic market-design treatment of an environment where agents do not use money but can signal their value via costly effort is, to my knowledge, absent.

With this motivation in mind, in this paper I study the welfare maximizing allocation of a set of heterogeneous indivisible objects to a number of heterogeneous risk-neutral agents. The objects are commonly ranked according to their quality. Agents are risk neutral, have unit demand, and independently distributed private valuations. Money is not available and the individual valuation represents the maximum cost that agents are willing to sustain in order to obtain an object of

E-mail address: dcond@essex.ac.uk.

[1] I thank Philippe Jehiel for advice throughout the project. I also thank Mark Armstrong, Gary Becker, Martin Cripps, Benny Moldovanu, Andras Niedermayer, Rakesh Vohra and Jidong Zhou for comments. The paper has benefited from the comments of an advisory editor and two referees.

[2] There is an extensive literature surveying non-market allocation mechanisms. For example, see Calabresi and Bobbitt (1978), Elster (1989, 1992), Okun (1975) and Walzer (1983).

[3] One obvious seminal reference is Vickrey (1961).

[4] See Schummer and Vohra (2007) for a survey on mechanism design without money.

0899-8256/$ – see front matter
doi:10.1016/j.geb.2012.02.018

unitary quality. The designer allocates the objects using observable information and extracts private information by setting up mechanisms whose outcome is conditional on the level of cost that agents are willing to sustain to secure an object.

The problem is non-trivial. Asking agents to sustain a cost to signal their values allows the designer to achieve a more efficient allocation, by assigning objects to those who value them the most. However, when the cost sustained is non-monetary and is wasted, screening generates a deadweight loss. Therefore, in contrast to models whit transferable utility, it might be optimal to assign the objects disregarding some private information.

The model above may be used to address several practical design problems. For example, consider the problem of administering immunization in case of an outbreak of pandemic flu. As in the recent H1N1 flu outburst, the number of available vaccines is likely to be fixed in the short run. Abstracting from external effects, the individual benefit of acquiring immunization will depend on a combination of public and private information. Moreover, there is consensus among public health specialists that scarce medical resources should not be awarded to the highest bidder. Instead, three types of mechanisms are recurrently proposed: waiting lines (i.e., an allocation based on the willingness to sustain a time-cost), lotteries and priority lists based on verifiable characteristics.[5] How to optimally design a mechanism to provide vaccination in case of emergency is an open problem. My work provides several insights on how to optimally balance the value of screening and its cost.

My results are potentially applicable also to other economic environments. Bagwell and Lee (2008) study money-burning advertising. Advertising is not informative but signals the quality of a firm to consumers. Hillman and Riley (1989) study lobbying as an all-pay auction. Lobbyists expend resources to secure a political prize but contributions are wasteful from a social standpoint. Baye et al. (2005) analyze legal systems. They argue that litigants increase their probability of winning a case by spending in legal services. Needless to say, spending in lawyers is socially wasteful. Taylor (1995) studies research tournaments. Contestants compete to find the innovation of highest value to the tournament sponsor. The winner receives a prize. The effort exerted by non-winning contestants is partially wasted. Applied to the environments above my analysis might provide insights on how to optimally regulate advertising, lobbying, the judicial system and research tournaments.

Preview of the results. My main contribution consists in the characterization of the set of ex-ante incentive efficient direct allocation mechanisms (Theorem 1).[6] A secondary contribution of this paper consists in showing how optimal direct mechanisms can be practically implemented using priority lists, lotteries, all-pay auctions, or other hybrid mechanisms (Theorem 2). A list of my main results follows.

Structure of the optimal mechanism. The optimal (incentive compatible) mechanism ranks agents in terms of their marginal contribution to social surplus. Objects are assigned, in order of decreasing quality, to agents who have higher ranking given their reported valuation. Remarkably, the ranking of agents is not affected by the relative scarcity of the objects, or by the extent of quality heterogeneity among them.

Optimality of priority lists and lotteries. If all the hazard rates of the prior distributions of values are monotonically increasing (e.g., values are drawn from a uniform distribution or an exponential), then an optimal mechanism does not extract any private information from agents, and takes the form of a priority list (i.e., an ordered list of the agents that have priority in the allocation), or a lottery if agents are ex-ante identical. Objects are allocated in descending order of quality to the agents which have (ex-ante) the highest expected values. The use of lotteries and priority lists, often in the form of point systems, is widespread. Conventional wisdom attributes their success to their fairness properties. A different rationale is provided here in terms of efficiency. These mechanisms prevent agents from engaging in wasteful rent seeking activities.

Optimality of screening. Full screening of private information is optimal if, and only if, all hazard rates are monotonically decreasing (e.g., values are drawn from a Pareto distribution or an exponential). When agents are symmetric, objects are assigned to the agents with highest realized values. A screening mechanism where the objects are assigned to the agents that sustain the highest cost implements the optimal mechanism.

Lotteries vs screening and the variability of the distribution of values. The optimality of lottery and screening is linked to the relative variability of the distribution of values and an exponential distribution with the same mean. In fact, a distribution with increasing (decreasing) hazard rate is characterized by a lower (higher) variance than an exponential distribution with the same mean.[7] To get an intuition, consider that when values are drawn from an exponential distribution both lottery and screening are optimal. Then, observe that for two distributions with the same mean the expected benefits from screening are smaller, and the costs higher, if the distribution is less variable. On one hand, it is less likely that two agents have different values – hence the efficiency gains from screening are smaller. On the other hand, it is more costly to screen individuals who are likely to have similar values.

Optimality of hybrid mechanisms. When hazard rates are not monotonic the optimal mechanism may require both screening and pooling of values. The optimal mechanism is a hybrid mechanism. For example, the optimal mechanism might give

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5 See, for example, the *Vaccine Prioritization Plan* – Supplemental Document A – prepared for the California Department of Health Services by the Berkeley Center for Infectious Disease Preparedness.

6 Since utility of agents is separable in the allocation and cost, once the problem has been set up as a mechanism design one, I can use the auction-design techniques developed in Myerson (1981).

7 An exponential distribution has a coefficient of variation (i.e., mean over standard deviation) equal to one. A distribution with increasing (decreasing) hazard rate has a coefficient of variation smaller (larger) than one (see Barlow and Proschan, 1965).

priority to a certain group of individuals and distribute the remaining objects using to the agents in the other group that are willing to sustain the highest cost.[8]

Lotteries vs screening under allocative and information externalities. Finally, I perform some comparative statics on the welfare generated by full screening and lotteries when the assumption of private values is relaxed. First, I show that, *ceteris paribus*, the introduction of positive (negative) externalities increases (decreases) the performance of screening in comparison to lotteries. On one hand, with positive (negative) externalities, the allocative-efficiency gain that is provided by a finer screening of values increases (decreases) compared to the no-externality case. On the other hand, the cost of providing such screening is reduced (increased). Second, I show that as the model tends toward a common value model, an equal chance lottery performs better than a contest. This result is most obvious in the common value case, where screening does not provide any welfare advantage.

Related literature. From a technical viewpoint this paper is closely related to McAfee and McMillan (1992) (henceforth MM). MM study optimal collusion without side-payments at first-price auctions. The weak-cartel problem is analogous to mine, because the money that the cartel members pay to the auctioneer is wasted. I generalize MM in several directions. I consider ex-ante asymmetric bidders. I do not impose restrictions on the hazard rates of the distributions of values. I consider the allocation of more than one object and allow for quality heterogeneity among objects. I span the entire ex-ante Pareto frontier. I discuss the implementation problem.

There are some recent papers that, independently, study optimal allocations in environments where agents compete by engaging in costly signaling: Chakravarty and Kaplan (2009), Hartline and Roughgarden (2008) and Yoon (2011). My results are significantly more general as I consider an environment with heterogeneous objects and asymmetric agents.

My work is also related to the analysis of sorting based on costly signals. The seminal reference is Spence (1973). More recently, Hoppe et al. (2009) (henceforth HMS) analyze matching between two sides of a market, with incomplete information. Both sides compete for better matches by engaging in costly signaling. HMS show that random matching performs better (worse) than assortative matching for one side when the distribution of types in that side has increasing (decreasing) hazard rate. My results strengthen this conclusion by proving that under increasing (decreasing) hazard rate random (assortative) matching is optimal among the class of all conceivable (i.e., incentive compatible) matching protocols. Hoppe et al. (2011) study coarse matching under incomplete information. In a coarse matching, agents on each side of the market are partitioned into exactly two classes. My work extends their conclusions by showing that coarse matching may maximize agents' welfare, outperforming both random and assortative matching.

Outline. Section 2 presents the model; Section 3 characterizes the optimal direct mechanism; Section 4 discusses practical implementation and Section 5 contains the comparative statics results.

2. The model

Agents and objects. Let $N = \{1, \ldots, n\}$ represent the set of agents. The set of objects is $M = \{1, \ldots, m\}$, with $m < n$. Objects are indivisible and are ranked in terms of some observable quality level. Let $0 < x_j < \infty$ indicate the quality of object j, and assume that $x_1 > x_2 > \cdots > x_m$, unless objects are said to be *identical* in which case assume that $x_1 = x_2 = \cdots = x_m = 1$.

Agent $i \in N$ has a *private valuation* $v_i \in V_i \equiv [0, \bar{v}_i)$, with $\bar{v}_i \leqslant \infty$. Henceforth, let $\boldsymbol{V} \equiv V_1 \times \cdots \times V_n$, $\boldsymbol{v} = (v_1, \ldots, v_n)$ and $\boldsymbol{v}_{-i} = \{v_j\colon\ j \in N \setminus i\}$. If agent $i \in N$ obtains object j and sustains a non-monetary cost equal to $c_i \geqslant 0$, his ex-post utility is $v_i x_j - c_i$.[9] If the agent obtains no object and pays c_i his utility is $-c_i$. Hence, $x_j v_i$ is the maximum cost that i is willing to expend to obtain object j.[10] Agents are risk neutral and have unit demand (i.e., if an agent obtains more than one object he consumes only the highest-quality one).[11]

Each agent possesses a set of *observable characteristics* which are common knowledge among all agents and known to the designer. Agent i's observable characteristics determine the beliefs that other agents, and the designer, hold about his private value. Observable characteristics are summarized by a continuous and strictly increasing cumulative distribution function F_i with support in V_i, density f_i, and finite expectation. Beliefs about individual values are treated as stochastically independent.

Designer. The task of the designer is to set up an allocation mechanism whose *equilibrium* outcome maximizes a weighted sum of the agents' ex-ante expected utilities, among all possible feasible mechanisms. As illustrated in Holmstrom and

8 As an example, consider the rationing of tickets for the Wimbledon Tennis Championship: part of the tickets is distributed using a lottery and part of the tickets is allocated using queues.

9 Assuming that utility is $v_i x_j - w_i(c_i)$ and that w_i is a commonly known and strictly increasing function such that $w_i(0) = 0$ would not provide more generality to the model.

10 Valuation is expressed with reference to some generic *cost* that the agent may be willing to sustain in exchange for the object. The type of cost that agents are willing to sustain will depend on the specific application (e.g., willingness to wait in line, exert an effort, sunk investment, etc.). What is important is that the cost is wasted and can be measured homogeneously across agents.

11 This valuation structure implies that if agent i is willing to sustain a larger cost than agent k to obtain object j, then i is also willing to sustain a larger cost than k to obtain any other object. However agent k might still be willing to sustain a cost for object j which is higher than what i would be willing to sustain for a different, lower quality, object.

Myerson (1983), a mechanism whose equilibrium maximizes a weighted sum of agents' ex-ante expected utilities is *ex-ante incentive efficient*.[12]

An *allocation mechanism* can be any game which agents play under incomplete information about their opponents' valuations. For each profile of actions an outcome of a mechanism consists of (i) a probability distribution over possible deterministic assignments of the m objects to the n agents and (ii) a vector of non-negative costs. Participation in the mechanism must be voluntary and comes at no cost.[13] Therefore, any *feasible* allocation mechanism must allow agents to opt out, thereby obtaining a payoff equal to zero.

3. Optimal direct mechanism design

Direct mechanisms. In this Section I appeal to the *revelation principle* and, without loss of generality, search for an optimal mechanism within the class of incentive compatible direct allocation mechanisms. A *direct allocation mechanism*, $\langle \boldsymbol{p}, \boldsymbol{c}\rangle$, is a mapping (i.e., a game) providing a distribution over all possible deterministic assignments of the objects and costs (i.e., an outcome) for each profile of reports from the agents about their private information (i.e., the actions). Given that the designer maximizes welfare, restricting attention to the case in which all objects are allocated is without loss of generality.

The multiple-objects problem is easier to deal with if the space of possible outcomes is rewritten using the Birkhoff–von Neumann theorem.[14] Hence, a direct allocation mechanism $\langle \boldsymbol{p}, \boldsymbol{c}\rangle$ can be expressed as a set of $n \times m$ allocation functions, $p_{i,j} : \boldsymbol{V} \to [0,1]$ for $i \in N$ and $j \in M$, and n cost functions, $c_i : \boldsymbol{V} \to \mathbb{R}^+$ for $i \in N$. The mechanism must be feasible, that is $\sum_{i=1}^n p_{i,j}(\boldsymbol{v}) = 1$ for all $j \in M$ and $\boldsymbol{v} \in \boldsymbol{V}$. Since agents have unitary demand it is never optimal to assign more than one object to any agent. Therefore, the designer restricts attention to mechanisms where $\sum_{j=1}^m p_{i,j}(\boldsymbol{v}) \leqslant 1$ for each $i \in N$ and $\boldsymbol{v} \in \boldsymbol{V}$.

Incentive compatibility. A direct allocation mechanism is *incentive compatible* if truthful reporting from all agents is an equilibrium and everyone obtains an expected payoff higher than zero.[15] This is formalized as follows. The ex-post utility to player i from announcing s_i when his true value is v_i, while all other players announce $\boldsymbol{v}_{-i}$ is

$$v_i\big[x_1 p_{i,1}(s_i, \boldsymbol{v}_{-i}) + \cdots + x_m p_{i,m}(s_i, \boldsymbol{v}_{-i})\big] - c_i(s_i, \boldsymbol{v}_{-i}).$$

Assuming that opponents report their valuations truthfully, the expected utility of agent i at the interim stage is:

$$U_i(v_i, s_i) = v_i \mathrm{E}_{\boldsymbol{v}_{-i}}\big[x_1 p_{i,1}(s_i, \boldsymbol{v}_{-i}) + \cdots + x_m p_{i,m}(s_i, \boldsymbol{v}_{-i})\big] - \mathrm{E}_{\boldsymbol{v}_{-i}}\big[c_i(s_i, \boldsymbol{v}_{-i})\big].$$

Therefore, a direct allocation mechanism $\langle \boldsymbol{p}, \boldsymbol{c}\rangle$ is *incentive compatible* if, and only if, for all $i \in N$ and $v_i \in V_i$:

$$U_i(v_i, v_i) = \max_{s_i \in V_i} U_i(v_i, s_i) \geqslant 0.$$

The next lemma offers a tractable characterization of the set of incentive compatible direct mechanisms. It is convenient to introduce an extra piece of notation and write $P_i(v_i) = \mathrm{E}_{\boldsymbol{v}_{-i}}[x_1 p_{i,1}(v_i, \boldsymbol{v}_{-i}) + \cdots + x_m p_{i,m}(v_i, \boldsymbol{v}_{-i})]$ and $C_i(v_i) = \mathrm{E}_{\boldsymbol{v}_{-i}}[c_i(v_i, \boldsymbol{v}_{-i})]$.

Lemma 1. *A direct allocation mechanism $\langle \boldsymbol{p}, \boldsymbol{c}\rangle$ is incentive compatible if, and only if, for all $i \in N$ and $v_i \in V_i$:*

$$\forall v_i' \in V_i: \quad v_i \geqslant v_i', \qquad P_i(v_i) \geqslant P_i(v_i') \quad \text{and} \tag{1a}$$

$$C_i(v_i) = v_i P_i(v_i) - \int_0^{v_i} P_i(x)\,dx. \tag{1b}$$

The proof is well known and it is omitted (see Myerson, 1981, for the one object case). Note that $C_i(0) = 0$ for all i, because $C_i(0) \leqslant 0$ is necessary for incentive compatibility, while $C_i(0) \geqslant 0$ follows since agents cannot receive positive transfers. One special cost rule that satisfies (1b) for any $\boldsymbol{p}$ is the well-known *Vickrey cost rule*:

$$c_i(\boldsymbol{v}) = v_i \sum_{j=1}^m x_j p_{i,j}(\boldsymbol{v}) - \int_0^{v_i} \left[\sum_{j=1}^m x_j p_{i,j}(z_i, \boldsymbol{v}_{-i})\right] dz_i. \tag{2}$$

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[12] It is ex-ante Pareto efficient within the set of incentive compatible mechanisms. No other incentive compatible mechanism can be found that makes everyone better off prior to the realization of private values.

[13] Assuming that agents incur a small fixed cost in order to participate in the allocation would not alter the conclusions of the analysis.

[14] The Birkhoff–von Neumann theorem states that any doubly stochastic matrix is the convex combination of permutation matrices. To apply the theorem assume that there are $n - m$ zero-quality objects.

[15] This definition of incentive compatibility embeds the individual rationality constraint imposed by the possibility of opting out from the mechanism.

According to the Vickrey rule only the winners of an object or those who participate in a lottery sustain a cost equal to the externality that they impose on other agents with their presence.[16] Under the Vickrey cost rule, reporting the true values is a (weakly) dominant strategy for all agents.

The objective function. There is a one to one mapping between the truthful equilibrium outcome of an incentive compatible direct mechanism and the direct mechanism itself. Therefore, we can state that a direct allocation mechanism $\langle \boldsymbol{p}, \boldsymbol{c}\rangle$ is ex-ante incentive efficient if it is incentive compatible, satisfies the feasibility and unit-demand constraints, and, for some set of non-negative Pareto weights $(w_1, \ldots, w_n)$, maximizes the following *welfare function*:

$$\mathrm{E}_{\boldsymbol{v}}\left\{\sum_{i=1}^{n} w_i\left[v_i \sum_{j=1}^{m} x_j p_{i,j}(\boldsymbol{v}) - c_i(\boldsymbol{v})\right]\right\}. \tag{3}$$

Absent incentive constraints a *first best* allocation assigns all objects to the agents with the highest realized values at no cost. Using the terminology of Becker (1973), the optimal matching of objects to agents under complete information is assortative.

It is a consequence of Lemma 1 that any first best allocation (i.e., an unconstrained ex-post efficient allocation) is not implementable, unless the allocation is dictatorial in a very strong sense.

Corollary 1. *A first best is not implementable unless $n-1$ agents have Pareto weight equal to zero. If the objects are identical, a first best allocation is not implementable unless $n-m$ agents have Pareto weight equal to zero.*

Proof. If the objects are not identical, a first best requires that the highest-quality object is allocated to the agent with highest weighted value. For any two agents i and k with positive w_i and w_k, choose two types t, t' belonging to both V_i and V_k in such a way that $w_i t' > w_k t$ and $w_k t' > w_i t$. These types always exist since V_i and V_k overlap in a neighborhood of 0. Hence, implementing the first best requires $C(t') > C(t)$. Thus, a first best cannot be achieved. When $x_1 = \cdots = x_j$, efficiency only requires assigning the m goods to the m agents with the highest values. The same logic applies to this case. $\square$

By using condition (1b) to substitute for the cost functions, changing the order of integration and integrating by parts, (3) can be written as:

$$\mathrm{E}_{\boldsymbol{v}}\left[\sum_{i=1}^{n} w_i \frac{1-F_i(v_i)}{f_i(v_i)} \sum_{j=1}^{m} x_j p_{i,j}(\boldsymbol{v})\right]. \tag{4}$$

The problem for the designer becomes that of maximizing (4) in $\boldsymbol{p}$ only, subject to (1a), feasibility and unit-demand constraints. A cost rule $\boldsymbol{c}$ satisfying (1b) can be obtained by the Vickrey rule in (2). I call an *optimal direct mechanism* a solution to the above problem.

To interpret the above objective function it is worthwhile to pursue the analogy with the optimal auction problem. Following Bulow and Roberts (1989), in the optimal auction problem the virtual valuation $v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ can be interpreted as the marginal revenue that a discriminating monopolist would obtain from selling to a certain bidder with a given value. Here, the inverse hazard rate $\frac{1-F_i(v_i)}{f_i(v_i)}$ of a given agent can be interpreted as the marginal consumer surplus generated from allocating the object to that agent.[17]

Optimal mechanisms. The building blocks for constructing an optimal direct mechanism under incomplete information are the *priority functions*, which assign to each agent a unique *priority level* as a function of his reported value. These functions are monotonic as a result of *ironing* of the inverse hazard rates.[18] Define, for all $x \in [0,1]$:

$$H_i(x) = \int_0^x \frac{1-z}{f_i(F_i^{-1}(z))}\,dz, \qquad G_i(x) = \mathrm{conv}\big\langle H_i(x)\big\rangle, \quad \text{and} \quad g_i(x) = \frac{dG_i(x)}{dx},$$

[16] For example, assume that there are three agents and two objects. Consider a particular realization $v_1 > v_2 > v_3$ and assume that the two objects are always assigned to the highest valued agents. Then, the payment of agent 1 will be $v_2(x_1 - x_2) + v_3 x_2$, the payment of agent 2 will be $v_3 x_2$, while the payment of agent 3 will be zero.

[17] To see the point consider the demand curve $q = 1 - F_i(v)$. Consumer surplus is $\int_0^q F_i^{-1}(1-x)\,dx - qF_i^{-1}(1-q)$. Marginal consumer surplus is $\frac{q}{f_i(F_i^{-1}(1-q))}$, that is $\frac{1-F_i(v)}{f_i(v)}$.

[18] See Myerson (1981). Ironing is needed when the inverse hazard rate is not monotone increasing in order to satisfy the monotonicity constraint implied by incentive compatibility.

where conv$\langle\cdot\rangle$ stands for the convex hull of the function, and where the right or left derivative of $G_i(v)$ is used when they do not coincide.[19] The *priority function* λ_i for agent i is:

$$\lambda_i(x) = w_i g_i\big(F_i(x)\big) \quad \text{for all } x \in V_i. \tag{5}$$

The next theorem characterizes an optimal mechanism.[20] The proof is in Appendix A.

Theorem 1. *An optimal direct mechanism* $\langle \boldsymbol{p}, \boldsymbol{c}\rangle$ *is defined as follows. The allocation rule* $\boldsymbol{p}$ *assigns objects in order of decreasing quality to the agents with highest reported priority* (*computed from* (5)) *until all objects are exhausted. Ties in priority are broken by an equal chance lottery. The cost rule* $\boldsymbol{c}$ *is defined by Eq.* (2) *for all* $i \in N$.

It is remarkable that the optimal mechanism does not depend directly on the scarcity of the objects (i.e., on $m - n$), nor on the level of heterogeneity among them (e.g., on the standard deviation of objects' quality). All types of agents are cardinally ranked, independently from other agents, only in terms of their marginal contribution to social surplus. Due to the quality-order of objects and the multiplicative structure of preferences, the ranking of agents' contributions to surplus is the same, no matter which object is allocated.

Pooling, screening and hybrid mechanisms. The rest of the section characterizes the optimal mechanism in more detail, providing some necessary conditions for pooling, screening, or hybrid mechanisms to emerge as optimal mechanisms. To avoid dealing with unnecessary details assume that $w_i > 0$ for all $i \in N$.

Corollary 2. *Suppose that all hazard rates of the distributions of values are monotonically non-decreasing. Then,* $\lambda_i(x) = w_i \mathrm{E}[v_i]$ *for all* $x \in V_i$.

Proof. H_i is concave. Hence G_i is a straight line from $H_i(0) = 0$ to $H_i(1) = \mathrm{E}[v_i]$. □

In this case the optimal mechanism is a *full pooling* mechanism, where no private information is extracted by the designer. The mechanism takes the form of a *priority list*. When all agents are *ex-ante symmetric* and have equal weighting, the optimal mechanism collapses to *equal chance lottery*. In the terminology of Becker (1973), the optimal matching is assortative ex-ante (i.e., conditional on observable characteristics), but random ex-post (i.e., when private information is taken into account).

Corollary 3. *Assume that hazard rates are all monotonically decreasing. Then priority functions are strictly increasing.*[21]

Proof. In this case $\lambda_i(x) = w_i \frac{1-F_i(x)}{f_i(x)}$ for all $x \in V_i$. In fact, G_i is convex and so $G_i(x) = H_i(x)$ for all $x \in [0,1]$. □

In this case the optimal mechanism is a *full screening mechanism*. If agents are *ex-ante identical* and weighted equally objects are always allocated to the agents with the highest *realized* values, with higher quality objects going to agents with higher valuations. In the matching terminology, assortative matching is optimal, both ex-ante and ex-post.

If agents are *ex-ante asymmetric* and the designer adopts equal weights, then the optimal mechanism will be biased in favor of the agents that appear to have the strongest claims. More precisely, the mechanism will favor those agents whose distribution hazard rate dominates those of the other agents.[22] This is to be contrasted with the revenue maximizing auction, where the designer discriminates in favor of the weakest bidder in order to extract higher payments from the strongest one.

Corollary 4. *Assume that the hazard rate of some agent* $i \in N$ *is monotonically increasing in an interval* $(v, v') \subset V_i$. *Then,* λ_i *is constant in* (v, v').

Proof. Note that H_i must be concave in $(F_i(v), F_i(v'))$. In fact, for z in $(F_i(v), F_i(v'))$, $H_i'(z) = \frac{1-F_i(F_i^{-1}(z))}{f_i(F_i^{-1}(z))}$ is decreasing. Since H_i is concave in $(F_i(v), F_i(v'))$, G_i will be a straight line in the same interval. It follows that $\lambda_i(\cdot)$ will be constant in (v, v'). □

19 $G_i(x)$ is the highest convex function such that $G_i(x) \leqslant H_i(x)$ $\forall x \in [0,1]$.

20 If attention is restricted to optimal mechanisms that treat ex-ante identical agents symmetrically, then there exists a unique optimal mechanism.

21 Distributions with monotone decreasing hazard rate include the exponential and the Weibull distributions with shape parameters below 1.

22 Agent i's hazard rate dominates that of agent j if his hazard rate is always lower than that of j. Under a monotonically decreasing hazard rate, we have that $\lambda_i(v) = \frac{1-F_i(v)}{f_i(v)}$ for all v. It follows that $\lambda_j(x) < \lambda_i(x)$ for any x. Therefore, i will be favored even if both i and j have the same value.

When the optimal mechanism assigns the same priority to all types in a given interval of the space of possible valuation, that interval is a *pooling region*. In general, when hazard rates are non-monotonic, the mechanism will be a *hybrid* one and contain some pooling and screening of values. The following example illustrates this case.[23]

Example. Consider the problem of distributing m identical tests for a rare but dangerous disease to a population of $n > m$ ex-ante identical agents, whose surplus is equally weighted by the designer. The disease can be successfully treated once discovered but is otherwise fatal. The occurrence of the disease is highly correlated with lifestyle. Therefore, potential individual benefits from taking the test, measured as the likelihood of having contracted the disease, depend on private information.[24] Individuals' values (for instance measured as the willingness to spend time in line) have been independently drawn from a *piecewise uniform bimodal* distribution:

$$f(v) = \begin{cases} \frac{7}{10} & \text{if } v \in [0,1], \\ \frac{1}{10} & \text{if } v \in (1,2], \\ \frac{2}{10} & \text{if } v \in (2,3]. \end{cases}$$

The optimal mechanism computed according to Theorem 1, assigns priority as follows:

$$\lambda_i(v) = \begin{cases} 0.65 & \text{if } v \in [0,1), \\ 1.16 & \text{if } v \in [1,3]. \end{cases}$$

Agents that declare a value below 1 obtain lower priority but are not required to sustain a cost, even if they obtain an object. Agents that declare a value above 1 get priority in the allocation, but they are required to sustain a positive expected cost. This cost must be such that an agent with value 1 is indifferent about declaring a value of 0 or a value of 1.[25]

4. Practical implementation

Aim of the section. This section illustrates how the outcome of an optimal direct mechanism can be implemented using appropriately designed *all-pay auctions*.[26] In a (first-price) all-pay auctions agents simultaneously submit a bid and the auctioneer assigns the objects based on the profile of bids. All bidders, regardless of whether they obtained an object or not, sustain a cost equal to their bid.

Practical implementation via all-pay auctions is especially relevant because all-pay auctions have been used to model a wide variety of economic interactions where agents engage in wasteful signaling activities. These include, to name a few, allocation through waiting lines (Holt and Sherman, 1982), lobbying (Hillman and Riley, 1989), R&D races (Che and Gale, 2003), competitions for a monopoly position (Ellingsen, 1991), and money-burning advertising (Bagwell and Lee, 2008).

All-pay implementation. An *all-pay auction* is an arbitrary function $\hat{\boldsymbol{p}}$ from $[0,\infty)^n$ (i.e., the profile of submitted bids) into $[0,1]^{n\times m}$ (i.e., an allocation of the objects), such that for all $\boldsymbol{b} \in [0,\infty)^n$, $i \in N$ and $j \in M$ we have $\sum_{i=1}^n \hat{p}_{i,j}(\boldsymbol{b}) = 1$ and $\sum_{j=1}^m \hat{p}_{i,j}(\boldsymbol{b}) \leqslant 1$. For each profile of bids $(b_1,\dots,b_n)$, the outcome of the all-pay action consists in the allocation of objects $\hat{\boldsymbol{p}}(\boldsymbol{b})$ (where $\hat{p}_{i,j}$ indicates the probability that object j is assigned to bidder i) and the profile of costs $b_1,\dots,b_n$ sustained by the agents.

The following theorem shows how to design $\hat{\boldsymbol{p}}$ to implement the same outcome of an optimal direct mechanism. The formulation is general and accommodate both full pooling and full screening mechanisms.

Theorem 2. *Let $\langle \boldsymbol{p}, \boldsymbol{c}\rangle$ be an optimal direct mechanism. For all $i \in N$, let*:

$$\hat{C}_i \equiv \{x \in [0,\infty) \colon C_i(v_i) = x \text{ for some } v_i \in V_i\}, \quad \text{and}$$

$$\hat{v}_i(b_i) = \begin{cases} C_i^{-1}(v_i) & \text{if } b_i \in \hat{C}_i, \\ \max\{x \in V_i \colon C_i(x) < b_i\} & \text{otherwise,} \end{cases}$$

where C_i^{-1} indicates an arbitrary selection from the inverse correspondence. For all i and j, define the all-pay auction *$\hat{\boldsymbol{p}}$ as follows*:

$$\hat{p}_{i,j}(b_1,\dots,b_n) = p_{i,j}\big(\hat{v}_i(b_i), \hat{v}_{-i}(b_{-i})\big).$$

The all-pay auction defined above has an equilibrium, with bidding strategies $b_i(v_i) = C_i(v_i)$ for $i \in N$, which is (interim) outcome equivalent to the truthful equilibrium of the optimal direct mechanism.

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[23] This example also shows that a coarse matching of agents to objects is in some cases optimal under non-monotone hazard rate. See McAfee (2002) for a discussion on the efficiency properties of coarse matching under complete information.

[24] In a more general heterogeneous agents formulation of this example the expected benefits may depend on a combination of private and public information.

[25] For example, if $n = 2$ and $m = 1$, the expected cost sustained by a player with a value above 1 must be set by the designer equal to: $\frac{2+\Pr\{v<1\}}{2+2\Pr\{v<1\}} = \frac{27}{34}$.

[26] A similar method can be used to implement the optimal mechanism using different payment rules.

Waiting-line auctions. The result above can be applied, for instance, to the design of waiting-lines auctions, in cases where agents are symmetric and object homogeneous. In a waiting-line auction agents simultaneously and independently decide when to join the line. Those who join the line earlier get higher priority and everyone, upon observing the queue, can leave the line at no cost. Lotteries break ties instantaneously. In my formulation the designer can tweak the waiting line by defining a set of pooling regions, thereby providing that all those who arrive in the interior of a given pooling region will all have equal priority.[27]

When the space of valuations is bounded, Corollary 4 implies that there will be always a pooling region in a neighborhood of the upper bound of the support.[28] This fact provides an interesting policy implication, given that for practical purposes a bound can almost always be placed on the set of possible valuations. Even in cases where using waiting-line auctions might be welfare enhancing over priority lists or lotteries, limiting the advance with which people can join a line, by giving the same priority to those who arrive earlier than a certain threshold, turns out to be always beneficial for welfare. While this observation seems at odds with empirical evidence, note that a cap on the line can always be implemented by limiting the amount of time that separates the announcement date from the distribution date.

5. Extensions

In the remainder of this section I investigate the consequences of relaxing the *private value* assumption on the relative welfare performance of the two polar mechanisms that emerged from the previous analysis, *full screening* and *full pooling*.

I restrict attention to the case where: (i) the m objects are identical (i.e., $x_j = 1$ for all $j \in M$); (ii) agents are ex-ante symmetric (i.e., for all $i \in N$, v_i is distributed, independently from $\boldsymbol{v}_{-i}$, according to a strictly increasing distribution F with support in $[0, \bar{v})$ and finite expectation); (iii) the designer treats the agents equally (i.e., $w_i = 1$ for all $i \in N$). Unless stated otherwise, the assumptions of Section 2 remain valid.

Allocative externalities. Assume that allocating an object to agent i provides him with a private benefit v_i and imposes on other agents an (additive) externality equal to αv_i.[29] Denote an outcome as $\{\tilde{p}_i, \tilde{c}_i\}_{i=1}^n$, where $\tilde{p}_i \in \{0, 1\}$ indicates whether or not an object is allocated to agent i, and $\tilde{c}_i$ represents the cost that i must sustain. The ex-post utility of i is:

$$\tilde{p}_i v_i + \alpha \sum_{j \in N/i} \tilde{p}_j v_j - \tilde{c}_i.$$

The externality parameter α can be either positive or negative, but must be small in absolute value: $1 > \alpha > -\frac{1}{n-1}$. If positive externalities are very high (i.e., $\alpha > 1$), then there is no conflict of interest within agents, since everyone prefers that the highest value agents obtain the objects. In this case, providing incentives comes at no cost and a first best can be implemented. Conversely, with large negative externalities (i.e., $\alpha < -\frac{1}{n-1}$) it is optimal not to allocate the objects at all.

Let's now consider full pooling and full screening outcomes. Full pooling is implemented by an equal chance lottery. The total welfare obtained by this mechanism is straightforward to compute. With full screening the m agents with the highest realized values obtain the objects. This outcome is implementable using a standard first-price auction. Using standard arguments to compute the welfare generated by the auction, I obtain the following proposition.[30]

Proposition 1.[31] *The ex-ante welfare generated by full screening is greater than or equal to the welfare generated by an equal chance lottery if and only if*:

$$\sum_{i=1}^{m} \mathrm{E}[v_{(i,n)} - v_{(m+1,n)}] + \alpha(n-1) \sum_{i=1}^{m} \big[\mathrm{E}[v_{(i,n)}] - \mathrm{E}[v]\big] + \alpha m \mathrm{E}[v_{(m+1,n)}] \geqslant m \mathrm{E}[v].$$

The first term on the left-hand side and the term on the right-hand side represent the expected welfare from screening and that from a lottery, with no externalities. Further, note that $\sum_{i=1}^{m} [\mathrm{E}[v_{(i,n)}] - \mathrm{E}[v]] \geqslant 0$, holding with equality only if $m = n$. Therefore, inspection of the equation above shows that the presence of positive externalities (i.e., $\alpha > 0$) tends to make, *ceteris paribus*, full screening more efficient than lotteries. Conversely, negative externalities tend to make lotteries more efficient than screening. The effect of externalities is always positive because $n > m$ and it is increasing in the scarcity of the object (i.e., it decreases as n approaches m).

[27] In this case the pooling region will have to be defined differently, given that the (observable) waiting line will be equivalent to a first-price rather than to an all-pay auction. An analogous implementation method can be applied.

[28] If the support of possible values is bounded, then the hazard rate of the distribution of values will be increasing in a neighborhood of the upper bound of the support (see Barlow et al., 1963).

[29] My qualitative conclusions would not change by assuming more general structure, where the externality on an agent j depends both on v_i and v_j.

[30] See Jehiel and Moldovanu (2000) and Das Varma (2002).

[31] I denote $v_{(m,n-1)}$ the random variable corresponding to the m-highest value out of a sample of $n-1$ independent draws from the distribution F. This notation is non-standard.

The result is fairly intuitive. Positive externalities increase the performance of screening over lotteries through a double channel. On one hand, the value of a more efficient allocation of objects to agents increases, because everyone benefits from having higher value individuals obtain an object. On the other hand, the cost of screening is reduced compared to the case with no externality. In fact, the expected cost of an agent who obtains an object is in line with the welfare loss that he imposes on the other individuals by participating in the allocation mechanism. The converse holds in the case of negative externalities, in which case a lottery accrues an advantage over screening.

Interdependent values. Next, I assume that each agent receives a private signal about his ex-post value from obtaining an object, but that the latter is positively influenced by the private signals that other agents receive.

I model interdependent values in the simplest possible way that allows me to embed both the common value and the private value case. Let $\alpha \in [0,1]$ and, as in the previous subsection, let $\{\tilde{p}_i, \tilde{c}_i\}_{i=1}^n$ indicate an outcome. The utility enjoyed by agent i is[32]:

$$\Big[\alpha v_i + (1-\alpha)\max_{j\in N} v_j\Big]\tilde{p}_i - \tilde{c}_i.$$

Also in this case full screening can be implemented using a standard first-price auction.[33] Comparing the expected welfare generated by screening with that produced by an equal chance lottery, I obtain the following proposition, whose proof is in Appendix A.

Proposition 2. *Assume that $\alpha > 0$. The ex-ante welfare generated by full screening is greater than or equal to the welfare generated by an equal chance lottery if and only if:*

$$\sum_{i=1}^{m} \mathrm{E}[v_{(i,n)}] - m\mathrm{E}[v_{(m+1,n)}]/\alpha \geqslant m\mathrm{E}[v].$$

When $\alpha = 1$ the inequality above compares the welfare from full screening with the welfare generated by an equal chance lottery in the standard private value case. Therefore, as α goes from one to zero and the model converges toward the common value case, full screening becomes, *ceteris paribus*, less advantageous than a lottery in terms of welfare.

The intuition is the following. Positive interdependencies in values reduce the benefits from screening without affecting its cost. This is most evident in the common value model, where $\alpha = 0$. An equal chance lottery always welfare dominates full screening because the distribution of the objects is welfare irrelevant, as long as all objects are allocated.

Appendix A

Proof of Theorem 1. First, rewrite the objective function using Lemma 1 to substitute for the cost functions:

$$\mathrm{E}_{\boldsymbol{v}}\left\{\sum_{i=1}^{n} w_i\left[v_i\sum_{j=1}^{m} x_j p_{i,j}(\boldsymbol{v}) - c_i(\boldsymbol{v})\right]\right\} = \sum_{i=1}^{n} w_i\left\{\int_0^{\bar{v}_i}\left(\int_0^{v_i} P_i(x)\,dx\right)dF_i(v_i)\right\}.$$

By changing the order of integration and integrating by parts:

$$\mathrm{E}_{\boldsymbol{v}}\left[\sum_{i=1}^{n} w_i \frac{1-F_i(v_i)}{f_i(v_i)}\sum_{j=1}^{m} x_j p_{i,j}(\boldsymbol{v})\right].$$

Assuming that the cost functions satisfy (1b), the designer's problem is now the following:

$$\max_{p_{i,j}:\boldsymbol{V}\to[0,1],\, i=1,\dots,n} \mathrm{E}_{\boldsymbol{v}}\left[\sum_{i=1}^{n} w_i \frac{1-F_i(v_i)}{f_i(v_i)}\sum_{j=1}^{m} x_j p_{i,j}(\boldsymbol{v})\right]$$

[32] My qualitative conclusions would continue to hold in a more general model where the value depends in more complex way on the signals of other agents.

[33] Bidding behavior in the first-price auction is characterized in Milgrom and Weber (1982).

subject to:

$$\sum_{i=1}^{n} p_{i,j}(\boldsymbol{v}) = 1 \quad \forall i = 1,\dots,n \text{ and } \boldsymbol{v} \in \boldsymbol{V},$$

$$\sum_{j=1}^{m} p_{i,j}(\boldsymbol{v}) \leqslant 1 \quad \forall j = 1,\dots,n \text{ and } \boldsymbol{v} \in \boldsymbol{V},$$

$$P_i(v) \geqslant P_i\big(v^*\big) \quad \forall i \in N,\ \forall v, v^* \in V_i\colon\ v \geqslant v^*.$$

It can be readily seen that the candidate solution satisfies the first two constraints above, and that $\boldsymbol{c}$ satisfies (1b). To prove that the third constraint (1a) is also satisfied, note that λ_i is the derivative of a convex function and therefore it is monotonically increasing. Then, $\forall \boldsymbol{v}_{-i}$, $\sum_{j=1}^{m} x_j p_{i,j}(\boldsymbol{v})$ is increasing in v_i, which implies that P_i is also increasing.

Now, let us sum and subtract, for each i, $P_i(v_i)\lambda_i(v_i)$ inside the objective function and rewrite it to obtain:

$$\sum_{i=1}^{n} \mathrm{E}_{v_i}\left\{P_i(v_i)\lambda_i(v_i) + P_i(v_i)\left[w_i \frac{1-F_i(v_i)}{f_i(v_i)} - \lambda_i(v_i)\right]\right\}.$$

Take the second term of this expression for every i:

$$w_i \int_0^{\bar v_i} P_i(v_i)\left[\frac{1-F_i(v_i)}{f_i(v_i)} - g_i\big(F_i(v_i)\big)\right] f_i(v_i)\,dv_i$$

integrating by parts:

$$w_i P_i(v_i)\big[H_i\big(F_i(v_i)\big) - G_i\big(F_i(v_i)\big)\big]\,\Big|_0^{\bar v_i} - w_i \int_0^{\bar v_i} \big[H_i\big(F_i(v_i)\big) - G_i\big(F_i(v_i)\big)\big]\,dP_i(v_i).$$

Take the first term of the expression above. It is equal to zero: $H_i(0) = G_i(0)$ and $H_i(1) = G_i(1)$, because G_i is the convex hull of the continuous function H_i and thus they coincide at endpoints. The continuity of H_i follows from assuming a strictly increasing F_i. The objective function becomes:

$$\sum_{i=1}^{n} \mathrm{E}_{\boldsymbol{v}}\left[\lambda_i(v_i)\sum_{j=1}^{m} x_j p_{i,j}(\boldsymbol{v})\right] - \sum_{i=1}^{n} w_i \int_0^{\bar v_i} \big[H_i\big(F_i(v_i)\big) - G_i\big(F_i(v_i)\big)\big]\,dP_i(v_i). \tag{6}$$

It is easy to see that the candidate solution $\langle \boldsymbol{p}, \boldsymbol{c}\rangle$ maximizes the first sum as it puts all the probability of higher valued objects on the players for whom $\lambda_i(v_i)$ is maximal.

To conclude the proof, I show that the second term of (6) is equal to zero. It must always be non-negative, as $\forall x \in [0,1]$ $H_i(x) \geqslant G_i(x)$. It is equal to zero because G_i is the convex hull of H_i and so, whenever $H_i(F_i(v_i)) > G_i(F_i(v_i))$, G_i must be linear. That is, if $G(x) < H(x)$, then $G_i''(x) = g_i'(x) = 0$. Therefore, to conclude, λ_i will be a constant in a neighborhood of v_i, which implies that P_i will also be a constant. □

Proof of Theorem 2. First, I show that if everyone follows the candidate equilibrium bidding strategy, then the outcome of the optimal direct mechanism is implemented. For any $i \in N$ and $v_i \in V_i$, suppose that the arrival strategy is $b_i(v_i) = C_i(v_i)$. Clearly, everyone is paying the same interim cost as in the optimal direct mechanism. Next, note that players who pay the same interim cost must have the same priority: for all $v, v' \in V_i$, if $C_i(v) = C_i(v')$, then $\lambda_i(v) = \lambda_i(v')$. Therefore, because the allocation is based on priorities only and fair lotteries break ties, types v and v' obtain the same ex-post allocation, for any $\boldsymbol{v}_{-i} \in \boldsymbol{V}_{-i}$. Therefore, the two allocations coincide.

Next, I show that all agents arriving at $C_i(v_i)$ is an equilibrium. We have seen that if agent i bids $C_i(v_i)$ and everyone else bids according to the candidate equilibrium strategy, i gets the same outcome of the optimal direct mechanism. Next, note that bidding outside $\hat C_i$ is dominated by bidding within $\hat C_i$ because the agents obtain the same priority that he could get by bidding strictly less. Finally, if everyone bids according to C_i, an agent i with value v_i faces the choice between bidding $C_i(v_i)$ or mimicking what an agent with some other value would choose, according to the candidate equilibrium strategy. Therefore, the payoffs are the same as in the direct optimal mechanism. It follows that, if the agent chooses to use a strategy other than $C_i(v_i)$, he will obtain the outcome assigned to an agent with a different value in the optimal direct mechanism. However, this is not possible because the optimal direct mechanism is incentive compatible.

Finally, we need to prove that feasibility is respected. That is, for every $b_1,\dots,b_n$ we must have that $\sum_i \hat p_{i,j}(b_1,\dots,b_n) \leqslant 1$. Observing that $(\hat v_i(b_i), \hat v_{-i}(b_{-i})) \in \boldsymbol{V}$ is sufficient to prove the claim, because feasibility is respected pointwise in the optimal mechanism. □

Proof of Proposition 1. Equilibrium utility in the auction for an agent with value v is:

$$G(v)\left(v-\frac{1}{G(v)}\int_0^v xg(x)\,dx\right)+\alpha\sum_{i=1}^m \mathrm{E}[v_{(i,n-1)}].$$

It follows that the total welfare generated by screening is:

$$\sum_{i=1}^m \mathrm{E}[v_{(i,n)}-v_{(m+1,n)}]+n\alpha\sum_{i=1}^m \mathrm{E}[v_{(i,n-1)}].$$

Next, note that the following recurrence relation holds for the moments of order statistics (for example, see David and Joshi, 1968):

$$n\mathrm{E}[v_{(x,n-1)}]=(n-x)\mathrm{E}[v_{(x,n)}]+x\mathrm{E}[v_{(x+1,n)}].$$

Therefore:

$$n\sum_{i=1}^m \mathrm{E}[v_{(i,n-1)}]=(n-1)\sum_{i=1}^m \mathrm{E}[v_{(i,n)}]+m\mathrm{E}[v_{(m+1,n)}].$$

With this in mind, the total welfare from screening can be rewritten as:

$$\sum_{i=1}^m \mathrm{E}[v_{(i,n)}-v_{(m+1,n)}]+\alpha(n-1)\sum_{i=1}^m \mathrm{E}[v_{(i,n)}]+m\alpha\mathrm{E}[v_{(m+1,n)}].$$

The utility from an equal chance lottery is:

$$m\mathrm{E}[v]+\alpha m(n-1)\mathrm{E}[v].$$

Comparing and rearranging one gets the desired result. □

Proof of Proposition 2. The welfare generated by full screening is greater or equal than the welfare from a lottery if:

$$\alpha\sum_{i=1}^m \mathrm{E}[v_{(i,n)}]+m(1-\alpha)\mathrm{E}[v_{(1,n)}]-m\mathrm{E}[v_{(m+1,n)}]\geqslant m\alpha\mathrm{E}[v]+m(1-\alpha)\mathrm{E}[v_{(1,n)}].$$

Comparing the two expressions, the term $m(1-\alpha)\mathrm{E}[v_{(1,n)}]$ cancels out. Dividing both sides by α and assuming that it is greater than zero delivers the desired result. □

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